COMPUTING THE CENTER OF AREA OF A CONVEX POLYGON*

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ABSTRACT. The center of area of a convex planar set X is the point p for which the minimum area of X intersected by any halfplane containing p is maximized. We describe a simple randomized linear-time algorithm for computing the center of area of a convex n-gon.

1 Introduction

Let X be a convex planar set with unit area. A *center of area* of X is a point p^* that maximizes the *cut off area function*

$$f(p) = \min\{\text{area}(h \cap X) \mid h \text{ is a halfplane that contains } p\} \ ,$$

and the value $\delta^* = f(p^*)$ is known as *Winternitz's measure of symmetry* [14]. The δ-level Γ_δ of X is defined as

$$\Gamma_\delta = \{p \mid f(p) = \delta\} \ .$$

It is known that Γ_δ is a simple closed convex curve and that Γ_{δ_1} is strictly contained in Γ_{δ_2} if $\delta_1 > \delta_2$. From this it follows that p^* is unique.

There is a long history of work on the center of area of convex sets. A classical result of Winternitz [3, pp. 54–55], which has been rediscovered many times [12, 16–19], is that $f(g) \geq 4/9$ where g is the centroid of X, with equality if and only if X is a triangle. (In d dimensions, Ehrhart [11] showed that $f(g) \geq d^d/(d+1)^d$ with equality if and only if X is a pyramid on any (d-1)-dimensional convex base.) For centrally symmetric sets, $f(g) = 1/2$, since any line through the point of symmetry cuts X into two pieces of equal area. Thus, $4/9 \leq f(g) \leq 1/2$ with $f(g) = 4/9$ for triangles and $f(g)$ close to $1/2$ for highly symmetric sets.

Although much is known about the center of area, it is quite nontrivial to determine the center of area for a given convex set. In a series of papers, Díaz and O'Rourke [7, 8, 9] develop an $O(n^6 \log^2 n)$ time algorithm for finding the center of area of a convex n-gon. The same authors give an approximation algorithm that runs in $O(GK(n+K))$ time, where G is the bit-precision of the input polygon P and K is the output bit-precision of the point p^*. Braß and Heinrich-Litan [4] describe an $O(n^2 \log^3 n\alpha(n))$ time algorithm for computing the center of area of a convex n-gon. As an application of tools for searching in arrangments of lines, Langerman and Steiger [15] present an $O(n \log^3 n)$ time algorithm for finding the center of area of a convex n-gon. All of these algorithms are deterministic.

In this paper we give a simple randomized linear-time algorithm for finding the center of area of a convex n-gon P, which also computes Winternitz's measure of symmetry for P. We proceed by first giving a linear-time algorithm for the decision problem: Does there exist a point p such that $f(p) > \delta$? We then

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apply a randomized technique due to Chan [5] to turn this decision algorithm into a linear-time optimization algorithm. For convenience, our model of computation is the real-RAM, though we do not use any functions that are specific to this model. We require only that it is possible to to compute the exact area of a convex polygon.

The remainder of the paper is organized as follows. Section 2 describes our algorithm for the decision problem and Section 3 shows how to convert this decision algorithm into an optimization algorithm. Section 4 summarizes and concludes with directions for future research.

2 The Decision Algorithm

In this section, we give an $O(n)$ time algorithm for the following decision problem: Is there a point p such that $f(p) \geq \delta$? An alternative statement of this problem is: is Γ_δ non-empty? In what follows, we show that Γ_δ can be computed in $O(n)$ time.

A δ-cut of P is a directed line segment uv with endpoints u and v on the boundary of P such that the area of P to the right of uv is at most δ. Note that, for any δ-cut uv, the point p can not be to the right of uv. On the other hand, if there is no δ-cut uv with p on its right, then $f(p) \geq \delta$. Therefore, each δ-cut defines a linear constraint on the location of p, which we call a δ-constraint. The answer to the decision problem is affirmative if and only if there is a point p that simultaneously satisfies all δ-constraints. If such a point p exists, we call the constraints feasible, otherwise we call them infeasible.

Unfortunately, every polygon has an infinite number of δ-cuts and hence an infinite number of δ-constraints. However, we will show that all constraints imposed by these δ-cuts can be represented succinctly as $O(n)$ non-linear (but convex) constraints that can be computed in $O(n)$ time.

To generate a representation of all δ-constraints, we begin by choosing a point u on the boundary of P and finding the unique point v so that uv is a δ-cut. Next, we sweep the points u and v counterclockwise along the boundary of P maintaining the invariant that uv has an area of exactly δ to its right. We continue this process until we return to the original points u and v.

Observe that, as long as u and v do not cross a vertex of P, the intersection of all δ-constraints belonging to an edge pair is a convex region whose boundary consists of at most 2 linear pieces and 1 non-linear piece. (See Figure 1.) In fact, this non-linear piece is a hyperbolic arc, since it consists of midpoints of δ-cuts. Furthermore, the description complexity of these pieces is constant, since they are defined by four vertices of P and one area. Thus, the intersection of all these δ-constraints can be computed explicitly in constant time. Since u and v sweep over each vertex exactly once, we obtain $2n$ such convex constraints whose intersection is equal to the intersection of all δ-constraints.

Therefore, the decision problem reduces to determining if the intersection of $2n$ convex regions is empty. We can compute an explicit representation of this intersection in $O(n)$ time, as follows: Separately compute the intersection of all δ-constraints that contain the point $(0, +\infty)$ and all δ-constraints that contain the point $(0, -\infty)$ and then compute the intersection of the two resulting convex regions. Since the δ-constraints are generated sorted by slope, the first step is easily done in $O(n)$ time using an algorithm similar to Graham's Scan [1, 13]. Since the two boundaries of the two resulting regions are x-monotone and upwards, respectively downwards, convex, their intersection (step two) can be computed in $O(n)$ time using a left-to-right plane sweep [2].

We have just proven:

Theorem 1. *There exists an $O(n)$ time algorithm for the decision problem: Does there exist a point p such that $f(p) \geq \delta$?*

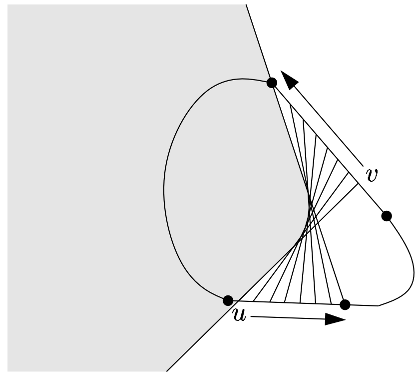

Figure 1: The intersection of all δ-constraints defined by a pair of edges makes a convex region whose boundary consists of at most 3 pieces.

3 The Optimization Algorithm

In this section, we show how to use the decision algorithm of the previous section along with a technique of Chan [5] to solve the optimization problem: What is the largest value of δ such that Γ_δ is non-empty? Chan's technique requires only that we be able to (1) solve the decision problem in $D(n) = \Omega(n^\epsilon)$ time, $\epsilon > 0$ and (2) generate a set of r subproblems each of size αn, $\alpha < 1$, such that the solution to the original problem is the minimum (or maximum) of the solutions to the subproblems. Under these conditions, the optimization problem can be solved by a randomized algorithm in $O(D(n))$ expected time.

To apply Chan's technique, we need a suitable definition of subproblem. Let S be a subset of edges of P. The S-induced δ-constraints are the set of all δ-constraints uv, where u and v are both on edges of S. The type of subproblems we consider are those of determining for a given set S and a value δ whether or not the S-induced δ-constraints feasible. To obtain a linear-time algorithm, we must be able to solve such subproblems in $O(|S|)$ time.

For a given set S, computing a representation of the S-induced δ-constraints, can be done using a modification of the sweep algorithm from the previous section so that it only considers δ-cuts uv where u and v are on elements of S. The only technical tool required for this modification is a data structure that, given two points u and v on elements of S (the boundary of P) tells us the area of P to the right of uv in constant time. This data structure is provided by Czyzowicz *et al* [6] who show that any convex n-gon can be preprocessed in $O(n)$ time so that the area of the polygon to the right of any chord uv can be computed in $O(1)$ time. Using this data structure, it is straightforward to generate a representation of S-induced δ-constraints in $O(|S|)$ time. Once we have computed these constraints, we can test if they are feasible in $O(|S|)$ time. Thus, Condition 1 required to use Chan's technique is satisfied with $D(n) = \Theta(n)$.

Next, we observe that Helly's theorem in the plane (c.f., Eckhoff [10]) implies that for any $\delta > \delta^*$ there exists a set of three δ-constraints whose intersection is empty. In our context, this means that P contains 6 edges such that, for any $\delta > \delta^*$, the δ-constraints induced by those edges are infeasible. Therefore, if a set S contains those 6 edges, then the S-induced δ-constraints are feasible if and only if $\delta \leq \delta^*$.

Therefore, we can solve our maximization problem as follows: Partition the edges of P in 7 groups, $E_1, \ldots, E_7$, each of size approximately $n/7$. Next, generate subsets $S_1, \ldots, S_7$, by taking all 7 6-tuples of $E_1, \ldots, E_7$. Note that, for each S_i, the S_i-induced δ-constraints are satisfiable if $\delta \leq \delta^*$, since they are just a subset of the original constraints. On the other hand, for the set S_j that contains the 6 edges guaranteed by Helly's theorem, the S_j-induced δ-constraints are not satisfiable for any $\delta > \delta^*$. Therefore,

$$\delta^* = \min \{\max \{\delta \mid S_i\text{-induced } \delta\text{-constraints are satisfiable}\} \mid 1 \leq i \leq 7\} \quad .$$

Finally, observe that each S_i is of size at most αn, for $\alpha = 6/7$. Therefore, we have satisfied the second condition required to apply Chan's optimization technique. This completes the proof of:

Theorem 2. *There exists a randomized $O(n)$ expected time algorithm for the optimization problem: What is the largest value δ^* for which Γ_{δ^*} is non-empty?*

Of course, once δ^* is known, an explicit representation of Γ_{δ^*} can be computed in $O(n)$ time. Alternatively, Chan's technique can also be made to output a point $p^* \in \Gamma_{\delta^*}$ [5].

4 Conclusions

We have given a randomized linear-time algorithm for determining the center of area of a convex n-gon. This algorithm is simple, implementable, and is asymptotically faster than any previously known algorithm.

Although our algorithm is simple and easy to implement, the constants are very large. A close examination of Chan's technique reveals that the number of subproblems generated in our application is actually $r \geq \binom{k}{6}$, where k is an integer that satisfies $\ln \binom{k}{6} + 1 < k/6$. The smallest such value of k is 146, which leads to $r = \binom{146}{6} = 12\,122\,560\,164$ subproblems. Reducing this constant while maintaining the $O(n)$ asymptotic running time remains an open problem. A linear-time deterministic algorithm is also an open problem.

Finally, we have not considered the problem of computing the center of area of a non-convex polygon. There are two different versions of this problem, depending on whether a cut is defined as a chord of P, which partitions P into two polygons, or a line which may partition P into many polygons. Approximation algorithms for the second case are considered by Díaz and O'Rourke [7]. To the best of our knowledge, there are no exact algorithms for either version.

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